The Complexity of Reachability in Affine Vector Addition Systems with States

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April 1, 2019

European Research Council
Established by the European Commission

The project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under grant agreement No 787367, from Discovery Grant from the Natural Sciences and Engineering Research Council of Canada (NSERC) and from the Fonds de recherche du Québec – Nature et technologies (FRQNT)

Overview

- Context: VASS, integer relaxations, and affine VASS
- State of the art: complexity of reachability in some $\mathbb{Z}$-A-VASS
- Setting: Classes of $\mathbb{Z}$-A-VASS
- Complexity of reachability for all classes of $\mathbb{Z}$-A-VASS
- Folklore is right about ($\mathbb{N}$-)A-VASS
- Proof components
- Conclusion and future work

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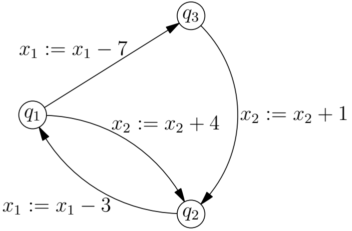
Vector Addition Systems (a.k.a. Petri Nets, etc.):

- Set of vectors (transitions) $V = \{\overline{v}_1, \ldots, \overline{v}_n\} \subset \mathbb{Z}^d$
- Trajectory: path $(\overline{x}_j)$ in $\mathbb{N}^d$ s.t. $\forall j : \overline{x}_{j+1} - \overline{x}_j \in V$

Vector Addition Systems with States

- Finite set Q of states
- Transitions: $q \xrightarrow{\overline{v}} q'$
- Trajectories: $(\overline{x}_j, q_j) \in \mathbb{N} \times Q$ s.t. $\forall j : q_j \xrightarrow{\overline{x}_{j+1} - \overline{x}_j} q_{j+1}$ is transition
- Same expressiveness as plain VAS



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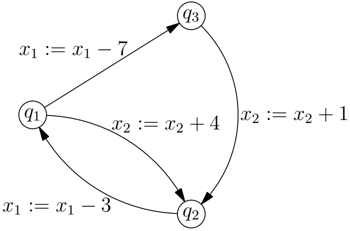

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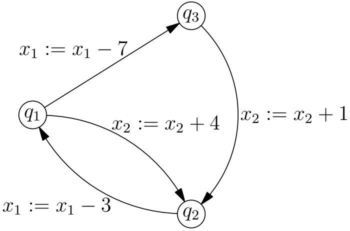

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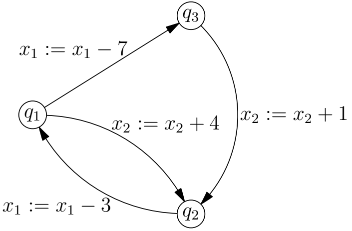

Classical setting for verification. . .

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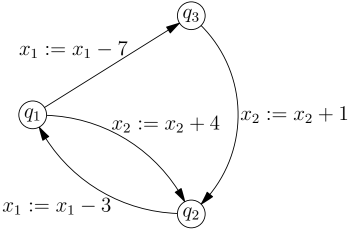

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- Trajectory now in $\mathbb{Z}^d \times Q$
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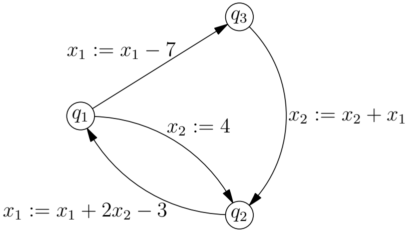

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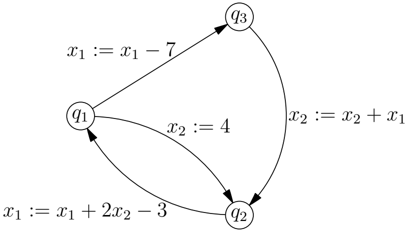

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- Affine VASS: undecidability, high expressiveness

Why not both?

- Still undecidable in general case [B., Haase, Mazowiecki 2018]

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$\overline{x} \mapsto A\overline{x} + \overline{b}$

What properties of A say about reachability problem complexity?

$\mathbb{Z}$-VASS extended with ...

- ... nothing (only identity matrix): **NP**-complete *[HH 2014]*
- ... resets (diagonal matrices in $\{0,1\}^{n \times n}$): **NP**-complete *[HH 2014]*
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- ... copies *or* with transfers: **PSPACE**-complete *[BHM 2018]*
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To use in modelling: characterise complexity for *all classes* of $\mathbb{Z}$-A-VASS?
So what is a class?

Resets: for single counter a_1: $a_1 := 0$;
 can apply to any one counter, the rest does not change
Copy: for two counters a_1, a_2: $a_1 := a_2$
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- **PSPACE**-complete
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Coincidence? No!

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Reset matrices: diagonal, with only 0 and 1 entries

Pseudo-transfer matrices: at most one non-zero element per column, elements 0, ±1

Pseudo-copy matrix: at most one non-zero element per row, elements 0, ±1

Theorem

The $\mathbb{Z}$-A-VASS reachability problem for matrix class $\mathcal{C}$ is:

- **NP**-*complete if $\mathcal{C}$ only contains reset matrices*
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  model Post Correspondence Problem

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- **PSPACE**-easy cases are closed classes with few matrices
- Positive results are all there
- Reductions needed
  (once the boundaries known!)

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Known:

- **NP**-completeness for resets
- **PSPACE**-completeness for copy/transfer/permutation
  guess memory contents, count guesses twice,
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Conclusion and future work

Achieved: classification of A-VASS and $\mathbb{Z}$-A-VASS reachability complexity for *classes of matrices*

What about individual instances of $\mathbb{Z}$-A-VASS?
(Work in progress) Arbitrary complexity at least between **P** and undecidable

What can be reachability complexity for given matrix monoid?
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Thanks for your attention!

Questions?