T. Golenishcheva-Kutuzova, A. Kazantsev,
Yu. Kudryashov, A. Kustarev,
G. Merzon, I. Yashchenko

Basics of Mathematics in Problems

with solutions
and comments

Part 1

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Contents

Introduction

In the mathematical classes of School #57, the well-know Moscow secondary school specializing in mathematics, besides lessons in algebra and geometry, there is another subject, traditionally called ‘mathematical analysis’. Unlike the lessons in other subjects, the lessons in that subject involve practically no explanations at the blackboard. Instead, the pupils are regularly given problem sheets, called *leaflets* (*listochki* in Russian[1]), which contain several problems, stated together with the necessary definitions.

The pupils solve and write down the solutions of the problems, each pupil at his/her own pace; there are no formal homework assignments, the pupil’s work is not marked (although twice a year an examination with marks is conducted); in class, each pupil individually discusses the solutions with one of the instructors – there is a team of 4-6 instructors at each lesson. It is those instructors who prepare the leaflets.

The present book contains all the leaflets given out to the ‘B’ class of Moscow School #57 that graduated in 2008, together with solutions and commentaries. This volume is Part I of the book. It includes the leaflets given out to the class when it was in the 8th form (out of 11). Additional problems are marked by a star (*), additional leaflets, by the letter ‘a’ added to the leaflet’s number.

About this introduction.

> Please, she asked, don’t finish telling the story. First let’s recall some details.
>
> Grigoriy Oster, *A Fairy Tale with Details*

Nobody ever reads long introductions. So we have decided to limit ourselves to a brief description of the teaching process in our class and a varied collection of details, in which the reader will perhaps find answers to possible questions.

On different approaches. We immediately warn the reader that different teams of instructors working with leaflets teach in different ways. So it makes little sense to speak of the ‘general approach to teaching math classes in teams’, and what follows is a description of how our own particular team worked and how we understand the teaching process.

[1]The English word ‘leaflet’ has a somewhat negative connotation, as in the expressions ‘political leaflets’, ‘advertising leaflets’, but the Russian word ‘listok’ does not bring to mind any negative associations.

What is more, it so happened that our team itself consisted of people of dissimilar temperaments, varying biases, different world outlook, diverse views on the teaching of mathematics[2].

During the teaching process, we often disagreed with each other and would spend a lot of time in heated discussions after classes. And, although most of the time none of us would change his/her position, the arguments of our colleagues often made us look at things from a different angle.

About our aims

> I candidly admit that in my long life I have never told my pupils anything about the 'meaning' of music; if there is such a thing, it has no need of me. Conversely, I always paid a great deal of attention to teaching my pupils to correctly count off eighths and sixteenths. Whether you are a teacher, a scientist, or a musician – revere 'meaning', but don't imagine that it can be taught.
>
> Herman Hesse, *The Glass Bead Game*

Let us say at once that our unique (or even our main) aim is not to bring up future professional mathematicians (although we try to give those pupils who aspire to become research mathematicians a chance to do that, provided they have the potential for it).

The things that we intend to teach the pupils can be divided into two groups. First of all – no matter how pretentious this may sound – we teach the pupils to think, to independently obtain new results, to experience mathematical discovery. If a pupil graduating from one of our math classes will never do any mathematics again, this experience will eventually help in one way or another.

On the other hand, since thinking and making mathematical discoveries is a complicated and creative activity, it is not clear how one should go about teaching such things. Therefore, during lessons, we are formally occupied with the second type of (much more modest) activity. We can say that we teach how to do four things: read, write, speak, and listen (the pupil *reads* the definitions and problems in the problem sheet, *writes* the solutions, *tells* them to the instructor, and *listens* to the instructor's comments – this is what we try to teach; but solving problems is something the pupils learn to do themselves.

We hope that such an approach to studying mathematics helps develop at least three skills, which are useful outside the class as well: 'the first is

[2]And this, in a sense, is not a bad thing – for instance, it gives the pupils the possibility to choose a preferred instructor to work with. Evidence of this diversity can probably be noticed in this book.

the ability to distinguish between truth and lies (lies understood in the mathematical sense, i.e., without intent to deceive); the second, to distinguish the meaningful from the meaningless; the third, to distinguish the comprehensible from the incomprehensible' (Vladimir Uspensky).

Besides all this, we would like our alumni to have some understanding of what mathematics is and how one works with it. This is useful not only to those who will be doing mathematics after leaving school, but also to those who will do no more mathematics – if only to make the latter group understand what it is, and abandon mathematics at the right time.

Finally, it just so happens that the classroom turns out to be the meeting place of youngsters who want to study mathematics and instructors who know and love the subject, and strive to share their knowledge. Perhaps it is the resulting communication which is the main goal of the process – just as it is in a musical society or a macramé club.

About the leaflet system

Leaflets. Mathematics is a creative activity; but there is no efficient *technology* for acquiring new mathematical knowledge. Now the only way to learn to swim is to try to swim in one way or another; looking at how others do it does not suffice. Similarly, the only way to learn how to make mathematical discoveries is practice: solving problems that provide the pupils with *new knowledge*.

Of course, this knowledge, its underlying facts have been known to humanity for a long time, but this isn't of much help to the pupils (only psychologically – in order to do something, it is useful to know that it can be done).

Incidentally, the last statement may be a bit misleading. The actual sequence of problems in each topic proposed to the pupils allows the pupils to move upwards, like up the steps of a staircase. To this end, the steps should be made high enough for the process to be interesting, but low enough so that each step is accessible to each pupil[3]. And such a step-by-step construction of each leaflet is based on the fact that the instructors know the solutions to the problems quite well.

At the same time, we include some difficult problems in the leaflets, sometimes even unsolved ones. The pupils will see that outwardly these

[3] Here we cannot avoid mentioning that the independent search for and choice of problems is an important part of a mathematician's work and it is *not* taught to the pupils in the framework of the leaflet system. Even the necessity of such an activity is hidden from the pupils working within that system – and this can give a distorted picture of how a research mathematician works.

problems appear to be of the same kind as the others, and try to solve them. It is pleasant to note that some pupils working within our system have obtained results worthy of publication in scientific journals[4].

Besides, each leaflet is an outline of sorts for a mathematical paper of the definition-theorem-proof kind, in which the pupil is asked to fill in the missing proofs. Thus each leaflet conveys an accepted method of structuring mathematical knowledge[5].

About the individual approach. We think that the idea of teaching a whole class according to the same program is counterproductive.

For that reason, besides the main program for all, there are additional leaflets about diverse topics (often diverging noticeably from the basic course); the pupils choose these additional leaflets according to their own taste. These leaflets (together with the additional problems in the required leaflets) compensate the difference in the pace of work of different pupils.

Besides different expectations and requirements imposed on the pupils, they are given different hints and tips, or, to the contrary, additional simplifying questions. These questions, together with the instructor's comments, fill in the gaps between the problems and the definitions in the leaflets, thereby creating (at least ideally) an individual course for each pupil.

Each instructor can only work in this way with small number of pupils whom he/she knows well enough. For this reason, the class needs several instructors, each one working with 3-5 fixed pupils. During the lesson, the instructors move around the classroom and periodically sit down next to one of 'their' pupils to discuss the problems. Usually twice a year a redistribution of pupils among the instructors occurs. Also, the pupils must pass a graded oral examination, which they never take with their own instructor.

About traditional methods. The main feature that distinguishes our approach from traditional lessons in school and the lecture-exercise class system at university is that we try to teach our pupils to discover something for themselves instead of following a given routine or using ideas explained by the teacher. It is precisely for this reason that we do not force the pupils to learn facts and prepared schemes by heart, and push them to invent new (for them!) methods of solution.

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[4]E.g., Yu. Makarychev. A short proof of Kuratowski's graph planarity criterion // J. of Graph Theory, 1997, Vol. 25, 129–131; A. Kustarev, Boundedness of finite vector sums and a proof of the Levi–Steinitz theorem // Mathematical Enlightment, Ser. 3, no. 7 [in Russian].

[5]The fact that this method is not the only one possible can easily be seen by comparing articles on the same subject in mathematics and physics.

Let us immediately point out that, at later stages, prepared expositions (books, lectures) are not only useful, but necessary. First of all, the study of any topic by solving problems requires a lot of time. Secondly, even if we assume that any topic can be expounded by means of a series of problems (which is not obvious), for most topics this has not been done (if only because this requires serious work by someone who has mastered the topic). Thus the idea that a sufficient (say, for doing serious mathematics) amount of knowledge can be obtained via the system of leaflets is not very realistic. Therefore, beginning with the 10th form, we give the pupils books to read and discuss, and organize lectures for them on certain topics.

But, at least at the beginning, the pupils must acquire a firm foundation of problems and theorems that they truly understand because they have discovered the proofs by themselves.

It should also be kept in mind that besides the analysis (calculus) course, School #57 always has courses in algebra and geometry[6] taught in a more traditional way.

About explanations at the blackboard. In forms 8–9, we gave explanations at the board only in two cases.

First, just before handing out a new leaflet, we sometimes explain the main ideas and motivations in an informal way, without stating rigorous theorems or going into the technical details of definitions; such explanations occur when there is no previous knowledge in the given topic.

Second, during the consultations that take place before each of the examinations, we present problem solutions, including the technical details. By that time the pupils have been working on the given topic for long enough and so can recognize problems that they may have spent a good deal of time thinking about.

About the examinations.

> – ...And was there a hole for the math examinations?
> – Sure, said Serpens as his eyes sparkled, ten elbows deep.
> – Just as for us! They put us in the hole, gave us problems, and those who couldn't solve them were never lifted out. I can tell you that the sight of the whitening bones of your predecessors really assists the mental process.
>
> Anna Korosteleva, *The Carmarthern School*

The examinations have several purposes.

[6]In particular, we recommend the brilliant geometry course of Rafail Gordin (who not only taught the traditional math courses in our class (8th 'B', but was also the class supervisor).

On one hand, they give the pupils the opportunity to understand what it is that they really know, and what they don't know, and this happens not only during the exam, but also in the preparation for it.

Generally speaking, preparation for the examination may be more useful than the exam itself. The approaching examination motivates (and that is another reason that we conduct it). During ordinary periods, the pupils have lots of other things to do – from strolling in the park to doing homework assignments in other subjects. While before the math exam, pupils concentrate on mathematics: preparation for the exam is a good occasion to recall what one has learned, to systematize that knowledge and to finally check out the fine points and problems from old leaflets that had not been solved.

To work all the time in such a regime is impossible – and that's why we conduct exams only twice a year – but doing it from time to time is very beneficial (it is impossible to walk slowly up an ice covered ridge, but one can make it to the top by running; in the same way, intense studies can yield a qualitative breakthrough only if they are interspersed with ordinary measured ones).

On the other hand, it is also interesting for us to find out what we have actually taught our pupils. Here the important thing is not how well they have 'mastered the material', and not even how well they have learned to solve problems (that is usually clear from everyday work in the classroom), but especially to find lacunas in the most unexpected places. Rather, what we are really interested in is their aptitude for mathematical communication (the instructor, working with the same pupil for a long period of time, can no longer objectively assess how well the pupil expresses his/her thoughts).

Finally, in forms 10-11 we invite to the examinations professional mathematicians, personal contacts with whom are interesting and beneficial to the pupils.

About the contents of the leaflets

On the choice of topics. The transmission of a maximal amount of knowledge is not one of our main goals; the concrete material that we choose is only a means, a convenient setting for the communication between pupils and instructors during the lessons. Hence the actual choice of topics is mainly motivated by the mathematical tastes of our team: it is always important to teach only the things that you like yourself.

In that context, we try to use topics that do not require too much preliminary knowledge; here we have in mind not only formally used definitions and theorems, but also facts needed to motivate the questions under study; insufficiently motivated and excessively abstract topics are badly assimilated by pupils in the 8th and 9th forms.

At the same time, a chosen topic must be sufficiently significant to ensure that its study will not reduce to a formal game with definitions. Otherwise, a situation arises in which an alumnus of a mathematics class knows a lot of fancy words, but is unable to prove or even to understand the proof of a nontrivial theorem.

Besides, we try to choose the topics so that the course will not be an incoherent collection of disconnected themes but will – at least to some extent – give an impression of *ascent*[7]. In our course for the 8th and 9th forms, the guideline is the construction of the real numbers: starting from basic set theory via the integers, rational numbers, ordered fields and on to calculus.

Finally, although the volume of acquired knowledge is secondary (as compared to acquiring skills in mathematical investigation), we try to include into the program a certain minimum without which the study of mathematics is impossible. For this reason, we sometimes hand out leaflets aiming at filling up lacunas in the pupil's knowledge. This is especially important at the beginning, when the pupils come to us with a completely different background of knowledge.

Writing the leaflets. It would seem that nothing is simpler: just take any sufficiently closed mathematical text (an article or chapter from a book) and copy from it the definitions, and state the lemmas and theorems in the form of problems (and possibly include a few additional intermediate lemmas). But it is clear that in this case all the comments that are formally not necessary for the proofs of the main results will be irredeemably lost. Thus, at a minimum, one must add (in the form of problems, possibly very easy ones, which simply fix certain assertions) examples and counterexamples demonstrating the necessity of assumptions in the theorems, as well as consequences of theorems showing their significance, and so on. As to the things that could not be set forth in this way – for instance, informal ideas and analogies – the instructor must keep them in mind in the discussion of the problems with the pupils; of course, this imposes definite requirements concerning the instructor's qualifications.

Let us say a few words about the composition of the leaflet. As the physicist Richard Feynman wrote, 'to understand means to get used to

[7]The majority of the leaflets combine into a more or less linear route, while the additional leaflets provide bifurcations in quite different directions.

and learn how to use.' And so each leaflet begins with sufficiently simple problems which allow the pupil to grasp the meaning of the basic concepts[8]. But of course one cannot learn mathematics by only solving simple problems, and so near the end of the leaflet the difficulty level of the problems increases, and in the longer leaflets two such difficulty peaks occur, one in the middle and the other at the end.

This composition, resembling an ascending staircase, allows the pupil to 'independently' obtain the proofs of significant theorems. Correspondingly (unlike the situation in the solution of technical exercises), the pupil can see the convincing result of his work, say that 'I have proved the fundamental theorem of arithmetic'. Here (in contrast with most olympiad problems), the obtained result is not only interesting *per se*, but is useful for what follows.

Of course, this type of outline of the leaflets imposes certain restrictions on the choice of material: since the size of the leaflet is limited[9], and each subsequent leaflet begins with easy problems, the effect of 'catching one's second breath' arises: such an ascending staircase never reaches the really difficult things, no matter how many leaflets are covered. This difficulty can be overcome (by the stronger pupils) thanks to the additional problems and extra leaflets (numbered 1, 2, … with letter 'a' for additional), the latter being longer and more difficult than the required leaflets, and, for all the pupils, by discussions with the instructor.

In conclusion of our discussion on the compilation of leaflets, we would like to warn against copying the leaflets from our course literally: on the one hand, they were written for a concrete group of pupils, and on the other hand, they reflect the mathematical tastes of concrete instructors. Nevertheless, we hope that this book will be useful for selecting the material to study in a math class.

On the axiomatic method and set theory. The road going up a mountain range is not direct and steep, it slowly winds along the slope of the mountain. The same is true in mathematics: beginning our course, we must first forget all the mathematics that was taught previously (our calculus course is formally self-contained: there are no references in the leaflets to any previous school material, and facts known from it cannot be used

[8]It sometimes happens that highly qualified instructors (especially when working with strong pupils) try to rapidly skim through problems that seem too simple and insufficiently meaningful; as a rule, this does not lead to good results.

[9]A ten page long mathematical paper is regarded as short, but a four page leaflet is so long that not every pupil will reach its end, and some of our colleagues believe that each leaflet must fit on one page – else it can no longer be called a 'leaflet'.

without proof) and we start from scratch, but at a different level[10]. In particular, on a different level of rigor: the course is based on the (informal) axiomatic method. The foundation on which the course is built consists of the undefined concepts of set and whole number. Our 'mathematical analysis' (calculus) course begins with an introduction to (naive) set theory.

To some extent this is a tribute to tradition, but it has its own reasons: this topic is usually new for our pupils, and this allows us to draw a clear line from the outset (which is very appropriate when the aims, the form, and the contents are completely changed); on new, easy to understand material it is simpler to specify the requirements concerning the rigor[11] of solutions and their written form. The choice of the axiomatic method as the foundation of our course was not the only possible one and, for us, it did not seem obvious. We do not advise following us in that choice before carefully assessing the *pros* and *cons*. And if one does begin the course in this way, it should be done accurately, taking into account the difference of level of the pupils: some have studied the subject in math circles and are ready for a higher degree of formalism, while others, if subjected to overly formal requirements, will lose all interest whatsoever in doing mathematics.

On cooperation and coercion

About mathematical discussions. From the very first lesson (and often before, in math circles), we try to show the pupils that we relate to them as colleagues, and so attempt to create an atmosphere of joint scientific work. This work usually consists in pupil and instructor jointly trying to assess the pupil's solution of a problem.

For such a relationship to be fruitful, we try to teach, from the very beginning, the *skills of mathematical dialogue* (which are valuable in themselves): to understand what is given, what must be proved, and what can be used in the process; to distinguish what has been proved from what hasn't; to coherently present one's thoughts, orally and in written form;

[10]And the amplitude increases as compared with the previous school course: we move down deeper, all the way to set theory, then wind up slowly (going through the integers and real numbers again) and end up at a much higher level.

[11]There is an opposite opinion about this, according to which, first of all, the solution of a problem (in this case, the axiomatic method as the solution to the problem of the foundations of mathematics) cannot be adequately understood unless the contents the problem (mathematics) are familiar, and, second, any method should be studied on significant examples, not on the simplest ones.

to state the negation of an assertion; to correct errors and fill up gaps in arguments. During the first lessons, most of the time is taken up by such apparently simple, but actually fundamental, things.

About writing down solutions. We work with pupils who usually think quite rapidly. This is wonderful and interesting, but the pupils usually think faster than they talk, and much faster than they write. And a lot of effort (and authority) is spent not only teaching them how to express thoughts on paper, but in actually convincing them that this is really necessary.

The main reason for insisting on this is that only when one begins to write out a solution does the structure of the argument become clear, and only then does one understand what is being said. A typical situation at the beginning of our studies is this: an 8th form student explains something and does not agree to write it down, saying that anyway it is *obvious*. The instructor then writes out what the pupil explained; the pupil sees that what is written is exactly what he/she said, but reading the text, exclaims: 'It seems I gave a correct solution, but what's written here is some kind of nonsense with lots of mistakes, it's basically incorrect'. To explain to him/her that there is an error in her oral argument is much harder, in particular because in response to the indication of an error the pupil can 'change the testimony' (and quite sincerely – in the process of a long conversation, it is difficult to remember what was said at the beginning), claiming that he/she didn't say that; besides, orally it is easier (consciously or not) to hide defects in the argument by means of rhetoric.

A solution written down on paper helps the pupil to structure his/her thoughts, to better grasp the logic of the argument (invented by the pupil), to follow through the whole chain of assertions. In particular, it is not unusual for students to find an error in their arguments.

About checking the solutions. Here it is important not to overdo the formal requirements of rigor to the detriment of substance. In practice, the verification of solutions always involves a competitive element: the pupil tries to convince the instructor that his/her argument is correct, while the instructor tries to find an error in it; if the instructor 'wins', then the pupil returns to the problem and tries to find an acceptable solution again. But we must not forget that the main goal of the instructor is not to find as many formal inaccuracies as possible, but to find out, together with the pupil, the gist of the matter. We would like each lesson to be a collaboration, not a competition.

In the converse case, – even leaving aside the psychological aspects of the situation, where at each lesson the pupil must compete with a person who is older and knows the subject matter much better – by the end of such studies, the pupil will reach the conclusion that mathematics reduces to formal manipulations with symbols according to fixed rules; for us, who definitely disagree with that viewpoint, this is something we would not like to happen.

About coercion. In our opinion, no teaching is possible without a certain amount of coercion. Those who believe that mathematics (as well as many other things) can be easily taught to a child in an atmosphere of happiness and love, are completely mistaken. But it is impossible to do creative work under the fear of punishment, so we must delicately use various means of compulsion, minimizing the more negative aspects.

The most important is to create an atmosphere in which it is *accepted* that studying and solving problems is a prestigious activity. Besides, we must create an atmosphere in which a pupil who has come to a lesson without any solved problems should feel uneasy meeting an instructor, as when meeting a colleague with whom you intended to work and discuss something, but came in vain, simply wasting other people's time.

In this context, we try to minimize the role of school marks, making the pupils understand that they are not working for formal grades (in the 8th form, many pupils, including some of the best ones, still believe that), but are working to solve problems, to appreciate the beauty of mathematics, and the highest prize is the pleasure of finding a solution, the esteem of colleagues (instructors and classmates) for the solution. Here the most important is the personal pleasure of finding a solution.

Clearly, teaching according to this approach is not efficient (or simply impossible) if the child doesn't like mathematics and doesn't want to study it. Incidentally, for this reason it is easy for us to screen our class from potential pupils whose parents, often friends of the instructors or of members of the school administration, pressure us to accept their siblings: we simply honestly explain that, as friends, the best we can do for the child is to protect him/her from life in such a penitentiary as our mathematics class.

About copying. The unpleasant situation when pupils copy solutions from classmates is basically eliminated if the instructors have enough patience and pedagogical skill to free the pupil from the psychological anxiety of getting a failing mark: we must work things out so that the pupils do not tend to laziness, but without punishing them for not submitting solutions, without visibly counting the number of solved problems. Otherwise, at that age it is very difficult not to succumb to the temptation of copying (and no development of creativity can then occur). We try to explain to the pupils (and to their parents) that the assessment of the results is not carried out by the formal count of the number of solved problems and that a successfully copied problem does not help the pupil to achieve the goals of our studies.

About the tempo. Due to the different initial levels of mathematical preparedness and different styles of thinking, our pupils solve problems at different speed, and we try to avoid any competition based on the formal number of solved problems. We immediately explain that we judge (both formally and informally) the individual work of each pupil, the intensity of his/her assiduity, on the basis of the pupil's possibilities at the given moment, and not in comparison to some fixed mean level.

Under this approach, the final marks given to the pupils are largely based on the subjective assessment of the instructor and makes no claim to objectivity. But in our practice, it usually turns out that the pupil's mark is a surprise to no one, and the pupils usually agree with the instructor's assessment.

As we mentioned before, there are no formal homework assignments, but the instructor indicates to the pupil (explicitly or implicitly) when it is time to finish working on the given leaflet (i.e., submit all the required problems in it). We try to work things out so that unfinished required leaflets do not accumulate, so we give out new leaflets only when the majority of pupils have worked out the old ones, and help out the lagging pupils.

The instructors

Students. The instructor in a math class does not necessarily have to be a mathematician, but it is important for that person to be interested in mathematics and know the subject. Actually, it turns out that the best instructors in a math class are undergraduate and graduate students in mathematics, who have recently graduated from a math class.

Such instructors feel closer to the pupils, there are no psychological barriers between them and their pupils (and so it is not surprising that the communication between them is not limited to lessons – there are short camping trips, campfire songs with a guitar, discussions about books and movies, and often this continues after graduation). These instructors have an overwhelming desire to share the knowledge recently acquired in school and at university. Finally, they still remember attitude from their own experience as school pupils *how* they were taught; and not only what worked, but what didn't. For this reason, they don't need any special pedagogical education, and so are ready to teach according to the given approach, provided that they are appropriately guided.

Such instructors usually constitute the majority of the team (it is for this reason that the specialized math school system is so stable[12]). This was so in our case.

The head of the team. When the atmosphere in the class is very informal, it is more difficult to maintain a reasonable level of discipline. The distinction between a creative atmosphere and total chaos is a very fine one. And when a critical mass of pupils who won't do any work is formed, the class falls apart: either the pupils openly stop doing anything, or an imitation of activity sets in (fortunately, this never happened in our classes).

The role of the leader (besides taking part in checking the solutions of problems) is, first of all, to have a correct feeling of what is going on in the class in general, and for each pupil in particular, and to accurately regulate the situation: praise one, reprimand another (without losing psychological contact), in some cases even change one of the instructors. Also, the head of the team must assess the level of the material, and choose the topics of study.

Of course, this choice is made collectively (not necessarily as the result of a discussion – on some topics the opinion of the team is unanimous), and in most cases the leader's opinion coincides with that of the rest of the team. In general, when things go well, the head of the team without being noticed works like the other instructors (it may even seem that no team leader is needed), but as soon as problems arise – he is the one who must make the appropriate decisions.

In conclusion – and this is most important, the team leader is the one who charges the pupils as well as his team with positive energy. And coming to each lesson, the leader must leave all his affairs and problems outside the classroom.

[12]That is what usually happens in efficiently working systems – their stability is due, to a great extent, to inertia: the students coming to teach in math schools believe that the way they were taught there is the most natural and correct one (modulo some small details), and don't bother to think about the reasons for deciding on the chosen approach, nor do they look for alternatives. Probably, we also are not entirely free of this point of view.

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